Mathematical Problems of Computer Science 52, 14–20, 2019.

UDC 519.1

Relative Lengths of Paths and Cycles in 2-Connected Graphs

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Abstract

Let l be the length of a longest path in a 2-connected graph G and c the circumference - the length of a longest cycle in G. In 1952, Dirac proved that $c > \sqrt{2l}$, by noting that "actually $c \geq 2\sqrt{l}$, but the proof of this result, which is best possible, is rather complicated". Let $L_1, L_2, ..., L_m$ be a vine on a longest path of G. In this paper, using the parameter m, we present a more general sharp bound for the circumference c including the bound $c \geq 2\sqrt{l}$ as an immediate corollary, based on elementary arguments.

Keywords: Longest cycle, Longest path, Circumference, Vine.

1. Introduction

We consider only undirected graphs with no loops or multiple edges. Let G be a 2-connected graph. We use c and l to denote the circumference (the length of a longest cycle) and the length (the number of edges) of a longest path of G. A good reference for any undefined terms is [1].

In 1952, Dirac [2] proved the following.

Theorem A: [2]. *In every 2-connected graph, $c > \sqrt{2l}$.*

In the same paper [2], Dirac considered a sharp version of Theorem A by noting that "actually $c \geq 2\sqrt{l}$, but the proof of this result, which is best possible, is rather complicated". Analogous questions were studied for k-connected graphs when $k \geq 3$ by Bondy and Locke ([4],[5]).

In this paper, using a new parameter, we present a more general sharp bound for the circumference c in 2-connected graphs in terms of l and the length of a vine on a longest path of G, including the bound $c \geq 2\sqrt{l}$ as a corollary, based on elementary arguments. In order to formulate this result, we need some additional definitions and notations.

The set of vertices of a graph G is denoted by $V(G)$ and the set of edges by $E(G)$. If Q is a path or a cycle, then the length of Q, denoted by $l(Q)$, is $|E(Q)|$ - the number of edges in Q. We write a cycle Q with a given orientation by $\overrightarrow{Q}$. For $x, y \in V(Q)$, we denote by

$x\overrightarrow{Q}y$ the subpath of Q in the chosen direction from x to y. We use $P = x\overrightarrow{P}y$ to denote a path with end vertices x and y in the direction from x to y. We say that vertex z_1 precedes vertex z_2 on $\overrightarrow{Q}$ if z_1, z_2 occur on $\overrightarrow{Q}$ in this order, and indicate this relationship by $z_1 \prec z_2$. We will write $z_1 \preceq z_2$ when either $z_1 = z_2$ or $z_1 \prec z_2$.

Let $P = x\overrightarrow{P}y$ be a path. A vine of length m on P is a set

$$\{L_i = x_i \overrightarrow{L}_i y_i : 1 \le i \le m\}$$

of internally-disjoint paths such that

(a) $V(L_i) \cap V(P) = \{x_i, y_i\}$ $(i = 1, ..., m)$,
(b) $x = x_1 \prec x_2 \prec y_1 \preceq x_3 \prec y_2 \preceq x_4 \prec ... \preceq x_m \prec y_{m-1} \prec y_m = y$ on P.

The main result is the following.

Theorem 1: *Let G be a 2-connected graph. If $\{L_1, L_2, ..., L_m\}$ is a vine on a longest path of G, then*

$$c \ge \begin{cases} \frac{2l}{m+1} + \frac{m+1}{2}, & \text{when } m \text{ is odd,} \\ \frac{2l-\frac{1}{2}}{m+1} + \frac{m+1}{2}, & \text{when } m \text{ is even.} \end{cases}$$

Equivalently, Theorem 1 can be formulated as follows, implying Dirac's conjecture as an immediate corollary.

Theorem 2: *Let G be a 2-connected graph. If $\{L_1, L_2, ..., L_m\}$ is a vine on a longest path of G, then*

$$c \ge \begin{cases} \sqrt{4l + (c-m-1)^2}, & \text{when } m \text{ is odd,} \\ \sqrt{4l + (c-m-1)^2 - 1}, & \text{when } m \text{ is even.} \end{cases}$$

Corollary 1: *In every 2-connected graph, $c \ge 2\sqrt{l}$.*

Note that if m is odd, then $c > 2\sqrt{l}$.

The following lemma guarantees the existence of at least one vine on a longest path in a 2-connected graph.

The Vine Lemma: [3]. *Let G be a k-connected graph and P a path in G. Then there are $k-1$ pairwise-disjoint vines on P.*

2. Proofs

Proof of Theorem 1. Let $P = x\overrightarrow{P}y$ be a longest path in G and let

$$\{L_i = x_i \overrightarrow{L}_i y_i : 1 \le i \le m\}$$

be a vine of length m on P. Put

$$L_i = x_i \overrightarrow{L}_i y_i \quad (i = 1, ..., m), \quad A_1 = x_1 \overrightarrow{P} x_2, \quad A_m = y_{m-1} \overrightarrow{P} y_m,$$

$$A_i = y_{i-1} \overrightarrow{P} x_{i+1} \quad (i = 2, 3, ..., m-1),$$

$$B_i = x_{i+1} \overrightarrow{P} y_i \quad (i = 1, ..., m-1),$$

$$l(A_i) = a_i \quad (i = 1, ..., m), \quad l(B_i) = b_i \quad (i = 1, ..., m-1).$$

Using the given vine $L_1, L_2, ..., L_m$, we construct a number of appropriate cycles and obtain a lower bound for the circumference as a mean of their lengths. First, we put

$$Q_0 = \bigcup_{i=1}^{m} (A_i \cup L_i),$$

$$Q_i = \bigcup_{j=i+1}^{m-i} (A_j \cup L_j) \cup B_i \cup B_{m-i},$$

where $i \in \{1, 2, ..., (m-1)/2\}$ when m is odd, and $i \in \{1, 2, ..., (m-2)/2\}$ when m is even. Since $l(L_i) \geq 1 \;\; (i = 1, 2, ..., m)$ and $a_1 \geq 1$, $a_m \geq 1$, we have

$$c \geq l(Q_0) = \sum_{i=1}^{m} l(L_i) + a_1 + a_m + \sum_{i=2}^{m-1} a_i \geq m + a_1 + a_m. \tag{1}$$

Case 1. m is odd.

For each $i \in \{1, 2, ..., (m-1)/2\}$, we have

$$c \geq l(Q_i) = b_i + b_{m-i} + \sum_{j=i+1}^{m-i} (a_j + l(L_j))$$

$$\geq b_i + b_{m-i} + \sum_{j=i+1}^{m-i} a_j + m - 2i. \tag{2}$$

By summing (1) and (2), we get

$$\frac{m+1}{2} c \geq \sum_{i=0}^{\frac{m-1}{2}} l(Q_i) \geq \sum_{i=1}^{m-1} b_i + \sum_{i=1}^{m} a_i + \frac{m+1}{2} m - 2 \sum_{i=1}^{\frac{m-1}{2}} i.$$

Since $l = \sum_{i=1}^{m-1} b_i + \sum_{i=1}^{m} a_i$, we have

$$\frac{m+1}{2} c \geq l + \frac{m+1}{2} m - \frac{m^2-1}{4} = l + \frac{(m+1)^2}{4},$$

implying that

$$c \geq \frac{2l}{m+1} + \frac{m+1}{2}.$$

Case 2. m is even.

As in Case 1, for each $i \in \{1, 2, ..., (m-2)/2\}$,

$$c \geq l(Q_i) \geq b_i + b_{m-i} + \sum_{j=i+1}^{m-i} a_j + m - 2i. \tag{3}$$

Case 2.1. $\frac{1}{2}c \geq b_{\frac{m}{2}}$.

By summing (1), (3) and $\frac{1}{2}c \geq b_{\frac{m}{2}}$, we get

$$\frac{m}{2}c + \frac{1}{2}c \geq \left(\sum_{i=1}^{m-1} b_i + \sum_{i=1}^{m} a_i \right) + \sum_{i=0}^{\frac{m-2}{2}} (m-2i)$$

$$= l + \frac{m}{2}m - 2\sum_{i=0}^{\frac{m-2}{2}} i = l + \frac{m(m+2)}{4},$$

implying that

$$c \geq \frac{2l - \frac{1}{2}}{m+1} + \frac{m+1}{2}.$$

Case 2.2. $\frac{1}{2}(c+1) \leq b_{\frac{m}{2}}$.

Put

$$R_0 = B_{\frac{m}{2}} \cup \bigcup_{i=1}^{\frac{m}{2}} (A_i \cup L_i),$$

$$R_m = B_{\frac{m}{2}} \cup \bigcup_{i=\frac{m+2}{2}}^{m} (A_i \cup L_i).$$

Further, for each $i \in \{1, 2, ..., \frac{m-2}{2}\}$, we put

$$R_i = B_{\frac{m}{2}} \cup B_i \cup \bigcup_{j=i+1}^{\frac{m}{2}} (A_j \cup L_j),$$

$$R_{m-i} = B_{\frac{m}{2}} \cup B_{m-i} \cup \bigcup_{j=\frac{m+2}{2}}^{m-i} (A_j \cup L_j).$$

Then clearly,

$$c \geq l(R_0) = b_{\frac{m}{2}} + \sum_{i=1}^{\frac{m}{2}} (a_i \cup l(L_i)) \geq b_{\frac{m}{2}} + \sum_{i=1}^{\frac{m}{2}} a_i + \frac{m}{2}, \tag{4}$$

$$c \geq l(R_m) = b_{\frac{m}{2}} + \sum_{i=\frac{m+2}{2}}^{m} (a_i \cup l(L_i))$$

$$\geq b_{\frac{m}{2}} + \sum_{i=\frac{m+2}{2}}^{m} a_i + \frac{m}{2}. \tag{5}$$

Furthermore, for each $i \in \{1, 2, ..., \frac{m-2}{2}\}$,

$$c \geq l(R_i) = b_{\frac{m}{2}} + b_i + \sum_{j=i+1}^{\frac{m}{2}} (a_j + l(L_j))$$

$$\geq b_{\frac{m}{2}} + b_i + \frac{m}{2} - i, \tag{6}$$

$$c \geq l(R_{m-i}) = b_{\frac{m}{2}} + b_{m-i} + \sum_{j=\frac{m+2}{2}}^{m-i} (a_j + l(L_j))$$
$$\geq b_{\frac{m}{2}} + b_{m-i} + \frac{m}{2} - i. \tag{7}$$

By summing (4), (5), (6) and (7), we get

$$mc \geq mb_{\frac{m}{2}} + \sum_{i=1}^{m} a_i + \left(\sum_{i=1}^{m-1} b_i - b_{\frac{m}{2}}\right) + m\frac{m}{2} - 2\sum_{i=1}^{\frac{m-2}{2}} i$$
$$= (m-1)b_{\frac{m}{2}} + \left(\sum_{i=1}^{m} a_i + \sum_{i=1}^{m-1} b_i\right) + \frac{m^2+2m}{4}$$
$$\geq \frac{(m-1)(c+1)}{2} + l + \frac{m^2+2m}{4},$$

implying that

$$c \geq \frac{2l}{m+1} + \frac{(m+2)^2-6}{2(m+1)} \geq \frac{2l-\frac{1}{2}}{m+1} + \frac{m+1}{2}.$$

Theorem 1 is proved. ■

Proof of Theorem 2. By (1), $c \geq m + a_1 + a_2 \geq m+2$. Let $c = m+y+2$ for some integer $y \geq 0$. By substituting $m = c-y-2$ in Theorem 1, we get

$$c \geq \sqrt{4l + (y+1)^2} = \sqrt{4l + (c-m-1)^2}$$

when m is odd; and

$$c \geq \sqrt{4l + y^2} = \sqrt{4l + (c-m-2)^2}$$

when m is even. Theorem 2 is proved. ■

To show the sharpness of the bounds in Theorems 1 and 2, let $P = x\overrightarrow{P}y$ be a path and let

$$\{L_i = x_i\overrightarrow{L}_i y_i : 1 \leq i \leq m\}$$

be a vine on P. Put

$$L_i = x_i\overrightarrow{L}_i y_i \quad (i=1,...,m), \quad A_1 = x_1\overrightarrow{P}x_2, \quad A_m = y_{m-1}\overrightarrow{P}y_m,$$
$$A_i = y_{i-1}\overrightarrow{P}x_{i+1} \quad (i=2,3,...,m-1), \quad B_i = x_{i+1}\overrightarrow{P}y_i \quad (i=1,...,m-1),$$
$$l(A_i) = a_i \quad (i=1,...,m), \quad l(B_i) = b_i \quad (i=1,...,m-1).$$

Let $y \geq 0$ by an integer and

$$a_1 = a_m = \frac{y}{2} + 1, \;\; a_2 = a_3 = ... = a_{m-1} = 0,$$

$$b_i = b_{m-i} = \frac{y}{2} + i + 1 \; (i = 1,2,..., \lfloor (m-1)/2 \rfloor).$$

If m is odd, then it is easy to see that

$$c = m + y + 2 = \frac{2l}{m+1} + \frac{m+1}{2} = \sqrt{4l + (c-m-1)^2}.$$

If m is even, we put $b_{m/2} = \frac{y}{2} + \frac{m+2}{2}$, implying that

$$c = m + y + 2 = \frac{2l - \frac{1}{2}}{m+1} + \frac{m+1}{2} = \sqrt{4l + (c-m-1)^2 - 1}.$$

Thus, the bounds in Theorems 1 and 2 are best possible.

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Submitted 25.05.2019, accepted 10.10.2019.

Շղթաների և ցիկլերի հարաբերական երկարությունների մասին 2-կապակցված գրաֆներում

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Ամփոփում

Դիցուք l-ը 2-կապակցված G գրաֆի ամենաերկար շղթայի երկարությունն է, իսկ c-ն՝ ամենաերկար ցիկլի երկարությունը: Դիրակը 1952-ին ցույց տվեց, որ $c > \sqrt{2l}$, միաժամանակ նշելով, որ իրականում $c \geq 2\sqrt{l}$, որը հնարավոր լավագույնն է, բայց այս գնահատականի ապացույցը բավականաչափ բարդ է: Դիցուք $L_1, L_2, ..., L_m$-ը G գրաֆի ամենաերկար շղթայի վրա մի բաղեղ է: Ներկա աշխատանքում m պարամետրի օգնությամբ բVրվում է մի ավելի ընդհանուր գնահատական, որտեղից

$c \geq 2\sqrt{l}$ գնահատականը բխում է որպես անմիջական հետևանք՝ հիմնված ոչ բարդ դատողությունների վրա:

Բանալի բառեր՝ ամենաերկար ցիկլ, ամենաերկար շղթա, ամենաերկար ցիկլի երկարություն, բաղեր:

Относительные длины цепей и циклов в 2-связных графах

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Аннотация

Пусть l обозначает длину длиннейшей цепи графа G, а с обозначает длину длиннейшего цикла. В 1952г. Дирак доказал, что $c > \sqrt{2l}$, отметив, что "в самом деле имеет место $c \geq 2\sqrt{l}$, что улучшить невозможно, но доказательство этой оценки достаточно сложно". Пусть $L_1, L_2, ..., L_m$ - плющ на длиннейшей цепи графа G. В настоящей работе приводится новая более общая оценка, откуда вытекает справедливость оценки $c \geq 2\sqrt{l}$ как непосредственное следствие, основанное на элементарных соображений.

Ключевые слова: длиннейший цикл, длинннейшая цепь, длина длиннейшего цикла, плющ.